*m*th LEVEL HARMONIC NUMBERS

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Abstract

We define m^{th} level harmonic numbers as a generalization of harmonic numbers. Then we construct the table of m^{th} level harmonic numbers which is like the Pascal's triangle. A formula for m^{th} level harmonic numbers containing binomial coefficients, as a generalization of Euler's formula for harmonic numbers, is also presented. From this formula, we also derive some relations between harmonic numbers and binomial coefficient.

m^{th} Level Harmonic Numbers

For a positive integer n, a harmonic number H_n is defined as $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Here we define m^{th} level harmonic number as follows:

$$H_n^{(0)} = 1;\ H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k} H_k^{(m-1)} \text{ for any positive integer } m.$$

Since $H_n = \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} \cdot 1 = \sum_{k=1}^{n} \frac{1}{k} H_k^{(0)} = H_n^{(1)}$, one can see that m^{th} level harmonic number is a generalization of a harmonic number.

Table of m^{th} Level Harmonic Numbers

From the definition of m^{th} level harmonic number, $H_n^{(0)} = 1$ and $H_1^{(m)} = 1$. For every $n \geq 2$ we have

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$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k} H_k^{(m-1)}$$

$$= \sum_{k=1}^{n-1} \frac{1}{k} H_k^{(m-1)} + \frac{1}{n} H_n^{(m-1)}$$

$$H_n^{(m)} = H_{n-1}^{(m)} + \frac{1}{n} H_n^{(m-1)}$$

From these facts we can construct the table of m^{th} level harmonic numbers like Pascal's triangle as follows:

n \ m	0	1	2	3	4
1	1	1	1	1	1
2	1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{31}{16}$
3	1	$\frac{11}{6}$	$\frac{85}{36}$	$\frac{575}{216}$	$\frac{1387}{491}$
4	1	$\frac{25}{12}$	$\frac{415}{144}$	$\frac{5845}{1728}$	$\frac{12456839}{3393792}$

In the above table, $H_n^{(m)}$ can be calculated as

$$\begin{array}{ccc} & & H_{n-1}^{(m)} \\ & & \downarrow \\ & & + \\ H_n^{(m-1)} & \xrightarrow{\times \frac{1}{n}} & H_n^{(m)} \end{array}.$$

m^{th} Level Harmonic Numbers and Binomial Coefficients

Euler's formula for harmonic numbers containing binomial coefficients is

$$H_n = H_n^{(1)} = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} \binom{n}{k}.$$

We will show that

$$H_n^{(m)} = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k^m} \binom{n}{k},$$

this formula can be seen as a generalization of Euler's formula for harmonic numbers.

Proof. We will prove by induction.

When $n=1$, $H_1^{(m)}=1=\sum_{k=1}^{1}(-1)^{k+1}\frac{1}{k^m}\binom{1}{k}$.

When $m=0$, $H_n^{(0)}=1$ and

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^0}\binom{n}{k}=\binom{n}{1}-\binom{n}{2}+\binom{n}{3}-\binom{n}{4}+\cdots+(-1)^{n+1}\binom{n}{n}=1.$$

Therefore $H_n^{(0)}=\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^0}\binom{n}{k}$.

Now we will show that the formula is true for $H_n^{(m)}$, $n\geq 2, m\geq 1$, by assuming that the formula is true for $H_{n-1}^{(m)}$ and $H_n^{(m-1)}$. Since $H_n^{(m)}=H_{n-1}^{(m)}+\frac{1}{n}H_n^{(m-1)}$, then

$$\begin{aligned}
H_n^{(m)} &= H_{n-1}^{(m)}+\frac{1}{n}H_n^{(m-1)}\\
&=\sum_{k=1}^{n-1}(-1)^{k+1}\frac{1}{k^m}\binom{n-1}{k}+\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^{m-1}}\binom{n}{k}\\
&=\sum_{k=1}^{n-1}(-1)^{k+1}\frac{1}{k^m}\frac{n-k}{n}\binom{n}{k}+(-1)^{n+1}\frac{1}{n\cdot n^{m-1}}\cdot 1+\sum_{k=1}^{n-1}(-1)^{k+1}\frac{1}{n\cdot k^{m-1}}\binom{n}{k}+\\
&=\sum_{k=1}^{n-1}(-1)^{k+1}\frac{1}{k^m}\frac{(n-k)+k}{n}\binom{n}{k}+(-1)^{n+1}\frac{1}{n^m}\\
&=\sum_{k=1}^{n-1}(-1)^{k+1}\frac{1}{k^m}\binom{n}{k}+(-1)^{n+1}\frac{1}{n^m}
\end{aligned}$$

$$H_n^{(m)}=\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^m}\binom{n}{k}$$

Harmonic Numbers and Binomial Coefficients

From the formula $H_n^{(m)} = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k^m} \binom{n}{k}$, one can derive some relations between harmonic numbers and binomial coefficient as follows:

$$\sum_{k=1}^{n} \frac{1}{k} H_k = H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2} (-1)^{k-1} \binom{n}{k}$$

$$\sum_{k=1}^{n} \frac{1}{k} (\sum_{i=1}^{k} \frac{1}{i} H_i) = \sum_{k=1}^{n} \frac{1}{k} H_k^{(2)} = H_n^{(3)} = \sum_{k=1}^{n} \frac{1}{k^3} (-1)^{k-1} \binom{n}{k}$$

$$\sum_{k=1}^{n} \frac{1}{k} H_k (H_n - H_{k-1}) = \sum_{k=1}^{n} \frac{1}{k} (\sum_{i=1}^{k} \frac{1}{i} H_i) = \sum_{k=1}^{n} \frac{1}{k^3} (-1)^{k-1} \binom{n}{k}.$$

Other formulas involving harmonic numbers and binomial coefficients can be found in [1, 2, 3] and others.

References

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